

# Optimality certificates for convex minimization and Helly numbers

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## Abstract

We consider the problem of minimizing a convex function over a subset of  $\mathbb{R}^n$  that is not necessarily convex (minimization of a convex function over the integer points in a polytope is a special case). We define a family of duals for this problem and show that, under some natural conditions, strong duality holds for a dual problem in this family that is more restrictive than previously considered duals.

## 1 Introduction

Insights obtained through duality theory have spawned efficient optimization algorithms (combinatorial and numerical) which simultaneously work on a pair of primal and dual problems. Striking examples are Edmonds' seminal work in combinatorial optimization, and interior-point algorithms for numerical/continuous optimization.

Compared to duality theory for continuous optimization, duality theory for mixed-integer optimization is still underdeveloped. Although the linear case has been extensively studied, see, e.g., [4, 5, 11, 12], nonlinear integer optimization duality was essentially unexplored until recently. An important step was taken by Morán et al. for conic mixed-integer problems [10], followed up by Baes et al. [2] who presented a duality theory for general convex mixed-integer problems. The approach taken by Moran et al. was essentially algebraic, drawing on the theory of subadditive functions. Baes et al. took a more geometric viewpoint and developed a duality theory based on lattice-free polyhedra. We follow the latter approach.

Given  $S \subseteq \mathbb{R}^n$  and a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we consider the problem

$$\inf_{s \in S} f(s). \quad (1)$$

We describe a geometric dual object that can be used to certify optimality of a solution to (1). For simplicity, let us consider the situation when the infimum of  $f$  over  $\mathbb{R}^n$  and over  $S$  is attained, and let  $x_0 \in \arg \inf_{x \in \mathbb{R}^n} f(x)$ . We say that a closed set  $C$  is an  *$S$ -free neighborhood of  $x_0$*  if  $x_0 \in \text{int}(C)$  and  $\text{int}(C) \cap S = \emptyset$ . Using the convexity of  $f$ , it follows that for any  $\bar{s} \in S$  and any  $C$  that is an  $S$ -free neighborhood of  $x_0$ , the following holds:

$$f(\bar{s}) \geq \inf_{z \in \text{bd}(C)} f(z) =: L(C), \quad (2)$$

where  $\text{bd}(C)$  denotes the boundary of  $C$  (to see this, consider the line segment connecting  $\bar{s}$  and  $x_0$  and a point at which this line segment intersects  $\text{bd}(C)$ ). Thus, an  $S$ -free neighborhood of  $x_0$  can be interpreted as a “dual object” that provides a *lower bound* of the type (2). As a consequence, the following is true.

**Proposition 1** (Strong duality). *If there exist  $\bar{s} \in S$  and  $C \subseteq \mathbb{R}^n$  that is an  $S$ -free neighborhood of  $x_0$ , such that equality holds in (2), then  $\bar{s}$  is an optimal solution to (1).*

This motivates the definition of a dual optimization problem to (1). For any family  $\mathcal{F}$  of  $S$ -free neighborhoods of  $x_0$ , define the  $\mathcal{F}$ -dual of (1) as

$$\sup_{C \in \mathcal{F}} L(C). \quad (3)$$

Assuming very mild conditions on  $S$  and  $f$  (e.g., when  $S$  is a closed subset of  $\mathbb{R}^n$  disjoint from  $\arg \inf_{x \in \mathbb{R}^n} f(x)$ ), it is straightforward to show that if  $\mathcal{F}$  is the family of *all*  $S$ -free neighborhoods of  $x_0$ , then strong duality holds, i.e., there exists  $\bar{s} \in S$  and  $C \in \mathcal{F}$  such that the condition in Proposition 1 holds. However, the entire family of  $S$ -free neighborhoods is too unstructured to be useful as a dual problem. Moreover, the inner optimization problem (2) of minimizing on the boundary of  $C$  can be very hard if  $C$  has no structure other than being  $S$ -free. Thus, we would like to *identify subfamilies  $\mathcal{F}$  of  $S$ -free neighborhoods that still maintain strong duality, while at the same time, are much easier to work with inside a primal-dual framework.* We list below three subclasses that we expect to be useful in this line of research. First, we need the concept of a *gradient polyhedron*:

**Definition 2.** *Given a set of points  $z_1, \dots, z_k \in \mathbb{R}^n$ ,*

$$Q := \{x \in \mathbb{R}^n : \langle a_i, x - z_i \rangle \leq 0, \quad i = 1, \dots, k\}$$

*is said to be a gradient polyhedron of  $z_1, \dots, z_k$  if for every  $i = 1, \dots, k$ ,  $a_i \in \partial f(z_i)$ , i.e.,  $a_i$  is a subgradient of  $f$  at  $z_i$ .*

We consider the following families.

- The family  $\mathcal{F}_{\max}$  of maximal convex  $S$ -free neighborhoods of  $x_0$ , i.e., those  $S$ -free neighborhoods that are convex, and are not strictly contained in a larger convex  $S$ -free neighborhood.
- The family  $\mathcal{F}_{\partial}$  of convex  $S$ -free neighborhoods that are also gradient polyhedra for some finite set of points in  $\mathbb{R}^n$ .
- The family  $\mathcal{F}_{\partial, S}$  of convex  $S$ -free neighborhoods that are also gradient polyhedra for some finite set of points in  $S$ .

We propose the above families so as to leverage a recent surge of activity analyzing their structure; the surveys [3] and Chapter 6 of [6] provide good overviews and references for this whole line of work. This well-developed theory provides powerful mathematical tools to work with these families. As an example, this prior work shows that for most sets  $S$  that occur in practice (which includes the integer and mixed-integer cases), the family  $\mathcal{F}_{\max}$  only contains polyhedra. This is good from two perspectives:

- polyhedra are easier to represent and compute with than general  $S$ -free neighborhoods,
- the inner optimization problem (2) of computing  $L(C)$  becomes the problem of solving finitely many continuous convex optimization problems, corresponding to the facets of  $C$ .

Of course, the first question to settle is whether these three families actually enjoy strong duality, i.e., do we have strong duality between (1) and the  $\mathcal{F}_{\max}$ -dual,  $\mathcal{F}_{\partial}$ -dual and  $\mathcal{F}_{\partial, S}$ -dual? It turns out that the main result in [2] shows that for the mixed-integer case, i.e.,  $S = C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$  for some convex set  $C$ , the  $\mathcal{F}_{\partial}$ -dual enjoys strong duality under conditions of the Slater type from continuous optimization. It is not hard to strengthen their result to also show that the  $\mathcal{F}_{\max} \cap \mathcal{F}_{\partial}$ -dual is a strong dual, under some additional assumptions.

In this paper, we give conditions on  $S$  and  $f$  such that strong duality holds for the dual problem (3) associated with  $\mathcal{F}_{\max} \cap \mathcal{F}_{\partial} \cap \mathcal{F}_{\partial, S}$ . Below we give an explanation as to why this family is very desirable. If these conditions on  $S$  and  $f$  are met, our result is stronger than Baes et al. [2]. For example, when  $S$  is the set of integer points in a compact convex set and  $f$  is any convex function, our

certificate is a stronger one. However, our conditions on  $S$  and  $f$  do not cover certain mixed-integer problems; whereas, the certificate from Baes et al. still exists in these settings. Nevertheless, it can be shown that in such situations, a strong certificate like ours does not necessarily exist.

**Definition 3.** A strong optimality certificate of size  $k$  for (1) is a set of points  $z_1, \dots, z_k \in S$  together with subgradients  $a_i \in \partial f(z_i)$  such that

$$Q := \{x \in \mathbb{R}^n : \langle a_i, x - z_i \rangle \leq 0, i = 1, \dots, k\} \text{ is } S\text{-free}, \quad (4)$$

$$\langle a_i, z_j - z_i \rangle < 0 \text{ for all } i \neq j. \quad (5)$$

Recall that  $a \in \partial f(z)$  if  $f(x) \geq f(z) + \langle a, x - z \rangle$  holds for all  $x \in \mathbb{R}^n$ . Since  $Q$  is  $S$ -free, for every  $s \in S$  there is some  $i \in [k]$  such that  $\langle a_i, s - z_i \rangle \geq 0$  and hence  $f(s) \geq f(z_i)$ . Thus, Property (4) implies that  $\min_{s \in S} f(s) = \min_{i \in [k]} f(z_i)$  holds. In other words, given a strong optimality certificate, we can compute (1) by simply evaluating  $f(z_1), \dots, f(z_k)$ . This implies that if a strong certificate exists, then the infimum of  $f$  over  $S$  is attained.

In order to verify that  $z_1, \dots, z_k$  together with  $a_1, \dots, a_k$  form a strong optimality certificate, one has to check whether the polyhedron  $Q$  is  $S$ -free. Deciding whether a *general* polyhedron is  $S$ -free might be a difficult task. However, Property (5) ensures that  $Q$  is *maximal*  $S$ -free, i.e.,  $Q$  is not properly contained in any other  $S$ -free closed convex set: Indeed, Property (5) implies that  $Q$  is a full-dimensional polyhedron and that  $\{x \in Q : \langle a_i, x \rangle = 0\}$  is a facet of  $Q$  containing  $z_i \in S$  in its relative interior for every  $i \in [k]$ . Since every closed convex set  $C$  that properly contains  $Q$  contains the relative interior of at least one facet of  $Q$  in its interior,  $C$  cannot be  $S$ -free.

For particular sets  $S$ , the properties of  $S$ -free sets that are maximal have been extensively studied and are much better understood than general  $S$ -free sets. For instance, if  $S = (\mathbb{R}^d \times \mathbb{Z}^n) \cap C$  where  $C$  is a closed convex subset of  $\mathbb{R}^{n+d}$ , maximal  $S$ -free sets are polyhedra with at most  $2^n$  facets [9]. In particular, if  $S = \mathbb{Z}^2$  the characterizations in [7, 8] yield a very simple algorithm to detect whether a polyhedron is maximal  $\mathbb{Z}^2$ -free.

In order to state our main result, we need the notion of the *Helly number*  $h(S)$  of the set  $S$ , which is the largest number  $m$  such that there exist convex sets  $C_1, \dots, C_m \subseteq \mathbb{R}^n$  satisfying

$$\bigcap_{i \in [m]} C_i \cap S = \emptyset \quad \text{and} \quad \bigcap_{i \in [m] \setminus \{j\}} C_i \cap S \neq \emptyset \text{ for every } j \in [m]. \quad (6)$$

**Theorem 4.** Let  $S \subseteq \mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function such that

- (i)  $\mathbb{O} \notin \partial f(s)$  for all  $s \in S$ ,
- (ii)  $h(S)$  is finite, and
- (iii) for every polyhedron  $P \subseteq \mathbb{R}^n$  with  $\text{int}(P) \cap S \neq \emptyset$  there exists an  $s^* \in \text{int}(P) \cap S$  with  $f(s^*) = \inf_{s \in \text{int}(P) \cap S} f(s)$ .

Then there exists a strong optimality certificate of size at most  $h(S)$ .

Let us first comment on the assumptions in Theorem 4. If  $\mathbb{O} \in \partial f(s^*)$  for some  $s^* \in S$ , then  $s^*$  is an optimal solution to (1), as well as to its continuous relaxation over  $\mathbb{R}^n$ . An easy certificate of optimality in this case is the subgradient  $\mathbb{O}$ . A quite general situation in which (ii) is always satisfied is the case  $S = (\mathbb{R}^d \times \mathbb{Z}^n) \cap C$  where  $C \subseteq \mathbb{R}^{d+n}$  is a closed convex set. In this situation, one has  $h(S) \leq 2^n(d+1)$ . The characterization of closed sets  $S$  for which  $h(S)$  is finite has received a lot of attention, see, e.g., [1]. Finally, note that (iii) implies that the minimum in (1) actually exists. As an example, (iii) is fulfilled whenever  $S$  is discrete (every bounded subset of  $S$  is finite) and the set  $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$  is bounded and non-empty for some  $\alpha \in \mathbb{R}$  (implying that the set is actually bounded for every  $\alpha \in \mathbb{R}$ ). This latter condition is satisfied, e.g., when  $f$  is strictly convex and has a minimizer. Another situation where (iii) is satisfied is when  $S$  is a finite set, e.g.,  $S = C \cap \mathbb{Z}^n$  where  $C$  is a compact convex set.

Also, if conditions (i) and (ii) hold, but (iii) does not hold, a strong optimality certificate may not exist. For example, consider  $S = \{x \in \mathbb{Z}^2 : \sqrt{2}x_1 - x_2 \geq 0, x_1 \geq \frac{1}{2}, x_2 \geq 0\}$  and  $f(x) = \sqrt{2}x_1 - x_2$ . In this case, no strong optimality certificate can exist, as the infimum of  $f$  over  $S$  is 0, but it is not attained by any point in  $S$ .

## 2 Proof of Theorem 4

We make use of the following observation. Let  $\text{conv}(\cdot)$  denote the convex hull and  $\text{vert}(P)$  denote the set of vertices of a polyhedron  $P$ .

**Lemma 5.** *Let  $S \subseteq \mathbb{R}^n$  and  $V \subseteq S$  finite such that  $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$ . Then we have  $|V| \leq h(S)$ .*

*Proof.* Let  $V = \{v_1, \dots, v_m\}$  and for every  $i \in [m]$  let  $C_i := \text{conv}(V \setminus \{v_i\})$ . Since  $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$ , we have  $C_i \cap S = V \setminus \{v_i\}$  for every  $i \in [m]$ . Thus,  $C_1, \dots, C_m$  satisfy (6) and hence  $m \leq h(S)$ .  $\square$

We are ready to prove Theorem 4. Let us consider the following algorithm (in fact, we will see that this is indeed a finite algorithm):

$$\begin{aligned}
Q_0 &\leftarrow \mathbb{R}^n, k \leftarrow 1 \\
\text{while } \text{int}(Q_{k-1}) \cap S \neq \emptyset : \\
t_k &\leftarrow \min\{f(s) : s \in \text{int}(Q_{k-1}) \cap S\} \\
C_k &\leftarrow \{x \in \mathbb{R}^n : f(x) \leq t_k\} \\
z_k &\leftarrow \text{any } s \in \text{int}(Q_{k-1}) \cap S \text{ with } f(s) = t_k \text{ such that } \dim(F_{C_k}(s)) \text{ is largest possible} \\
a_k &\leftarrow \text{any point in } \text{relint}(\partial f(z_k)) \\
Q_k &\leftarrow \{x \in Q_{k-1} : \langle a_k, x - z_k \rangle \leq 0\} \\
k &\leftarrow k + 1
\end{aligned} \tag{7}$$

In the above,  $\text{relint}(\cdot)$  denotes the relative interior and  $\dim(\cdot)$  the affine dimension. For a closed convex set  $C \subseteq \mathbb{R}^n$  and a point  $p \in C$  we denote by  $F_C(p)$  the smallest face of  $C$  that contains  $p$ .

Remark that iteration  $k$  of the algorithm can always be executed, as the set  $Q_k$  is a polyhedron and hence by the assumption in (iii) the minimum in (7) always exists. Furthermore, since  $a_k \in \text{relint}(\partial f(z_k))$  we have

$$F_k := F_{C_k}(z_k) = \{x \in C_k : \langle a_k, x - z_k \rangle = 0\} \tag{9}$$

*Claim 1:* For every  $k$  we have that  $\langle a_i, z_j - z_i \rangle < 0$  holds for all  $i, j \leq k$  with  $i \neq j$ .

Let  $k \geq 2$  and assume that the claim is satisfied for all  $i, j \leq k-1$ ,  $i \neq j$ . Since  $z_k \in \text{int}(Q_{k-1})$  and  $a_i \neq 0$  by assumption (i), we have that  $\langle a_i, z_k - z_i \rangle < 0$  for every  $i < k$ .

It remains to show that  $\langle a_k, z_i - z_k \rangle < 0$  for every  $i < k$ . Since  $a_k \in \partial f(z_k)$ , we have that  $\langle a_k, z_i - z_k \rangle \leq f(z_i) - f(z_k)$  and for  $i < k$  by (7) we have  $f(z_i) \leq f(z_k)$ . Therefore  $\langle a_k, z_i - z_k \rangle \leq 0$  and if  $\langle a_k, z_i - z_k \rangle = 0$ , then  $f(z_i) = f(z_k)$ . Assume this is the case. Since  $\langle a_i, z_k - z_i \rangle < 0$  we have  $z_k \notin F_i$  and in particular

$$F_i \neq F_k. \tag{10}$$

By (9) this means that  $z_i \in F_k$  holds. Since  $F_i$  is the smallest face that contains  $z_i$ , this implies  $F_i \subseteq F_k$ . By (8), we have that  $\dim(F_i) \geq \dim(F_k)$  and thus  $F_i = F_k$ , a contradiction to (10).

*Claim 2:* For every  $k$  we have that  $V := \{z_1, \dots, z_k\}$  satisfies  $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$ .

It is easy to see that Claim 1 implies  $V = \text{vert}(\text{conv}(V))$ . For the sake of contradiction, assume there exists some  $s \in (\text{conv}(V) \setminus V) \cap S$ . By Claim 1, we have  $s \in \text{int}(Q_k)$ . Therefore by (7) we have  $f(s) \geq t_k$ . Since  $f$  is convex and  $s \in \text{conv}(V)$ , this implies  $f(s) = t_k$ . Let  $a \in \text{relint}(\partial f(s))$  and

consider  $F := F_{C_k}(s) = \{x \in C_k : \langle a, z_i - s \rangle = 0\}$ . Since  $V \subseteq C_k$ , we have that  $z_i \in F$  for at least one  $i \in [k]$ . Due to  $\langle a, z_i - s \rangle \leq f(z_i) - f(s)$  we must have  $f(z_i) = t_k$  and hence  $F_i \subseteq F$ . By (8), we further have  $\dim(F_i) \geq \dim(F)$ , which shows  $F_i = F$ . However, by Claim 1 we have  $z_j \notin F_i$  for all  $j \neq i$  and hence  $s \notin F_i$ , a contradiction since  $s \in F$ .

*Claim 3: The algorithm stops after at most  $h(S)$  iterations and  $Q := Q_k$  is  $S$ -free.*

Note that the set  $V := \{z_1, \dots, z_k\}$  becomes larger in every iteration. By Claim 2 and Lemma 5 we must have  $k \leq h(S)$  and hence the algorithm stops after at most  $h(S)$  iterations. Since the algorithm stops if and only if  $Q_k$  is  $S$ -free, this proves the claim.  $\square$

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